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New scale-relativistic derivations of Pauli and Dirac equations

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Abstract
In scale relativity, quantum mechanics is recovered by transcribing the classical equations of motion to fractal spaces and demanding, as dictated by the principle of scale relativity, that the form of these equations be preserved. In the framework of this theory, however, the form of the classical energy equations both in the relativistic and nonrelativistic cases are not preserved. Aiming to get full covariance, i.e., to restore to these equations their classical forms, we show that the scale-relativistic form of the Schrödinger equation yields the Pauli equation, whilst the Pissondes's scale-relativistic form of the Klein–Gordon equation gives the Dirac equation.

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1. Introduction

Taking as its basic assumption the nondifferentiability of space, Nottale's scale relativity permits us to obtain the Schrödinger equation of a microscopic particle from the generalization to a fractal space of the classical equation of motion of this particle [1–3]. Then, assuming the nondifferentiability of the whole spacetime, Nottale showed that the Klein–Gordon equation of a relativistic microscopic particle is also recovered from the generalization to a fractal spacetime of the relativistic equation of motion of this particle [2, 4, 5]. More recently, Célérier and Nottale obtained, within the framework of this theory, both the Dirac equation [6, 7] and the Pauli equation [8].

To derive the Schrödinger equation, Nottale considered a symmetry breaking of the reflection invariance of the differential-time element ($\mathrm{d}t \longrightarrow -\mathrm{d}t$), whereas for the derivation of the Klein–Gordon equation he considered the symmetry breaking of the reflection invariance of the differential proper-time element ($\mathrm{d}s \longrightarrow -\mathrm{d}s$). To get the Dirac and Pauli equations, however, Célérier and Nottale introduced additional symmetry breakings; namely, the breaking of the symmetries ($\mathrm{d}x^\mu \longrightarrow -\mathrm{d}x^\mu$) and ($x^\mu \longrightarrow -x^\mu$). In doing so, they showed that the

Klein–Gordon equation may be written in a bi-quaternionic form, which in turn leads to the Dirac equation [6, 7]. In [8], the Pauli equation is recovered as a non-relativistic limit of the Dirac equation in the quaternionic formalism.

In the present work, we shall see that it is possible to obtain these last two equations without these additional symmetry breakings, by simply using the scale-relativistic forms of the Schrödinger and the Klein–Gordon equations. We begin in section 2 by briefly reviewing Nottale's derivation of the Schrödinger and Klein–Gordon equations. In section 3, we give a new derivation of the Pauli equation. In section 4, we give a new derivation of the Dirac equation. Section 5 contains some concluding remarks on this work.

2. The Schrödinger and Klein–Gordon equations

In scale relativity, space is fractal and nondifferentiable, physical quantities are resolution-dependent, and the local differential-time reflection invariance ($\mathrm{d}t \longrightarrow -\mathrm{d}t$) is broken [1, 3]. That is, at each point $x(t, \mathrm{d}t)$ of space corresponds two derivatives, i.e., two velocities,

$$\frac{\mathrm{d}}{\mathrm{d}t_+}x(t,\mathrm{d}t) = \frac{x(t+\mathrm{d}t,\mathrm{d}t) - x(t,\mathrm{d}t)}{\mathrm{d}t} = V_+[x(t), t, \mathrm{d}t], \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t_-}x(t,\mathrm{d}t) = \frac{x(t,\mathrm{d}t) - x(t-\mathrm{d}t,\mathrm{d}t)}{\mathrm{d}t} = V_-[x(t), t, \mathrm{d}t]. \tag{2}$$

(The resolution here is identified with the differential element $\mathrm{d}t$.) By taking what is called the classical or the 'large-scale' parts of these quantities [6, 7, 9], one is left with two scale-independent velocities, v_+ and v_-. Also, the existence of two kinds of derivatives yields two total derivatives with respect to time of any fractal function in the fractal space:

$$\frac{\mathrm{d}f}{\mathrm{d}t_\pm} = \left(\frac{\partial}{\partial t} + v_\pm \cdot \nabla \pm \mathcal{D}\Delta\right) f, \tag{3}$$

where $\mathcal{D}$ is a parameter characterizing the fractal behaviour of trajectories and measuring the amplitude of the fractal fluctuations [9, 10]. Combining the two velocities in one hand and the two total derivatives on the other, Nottale defined a complex velocity or a 'bi-velocity'

$$\mathcal{V} = \frac{v_+ + v_-}{2} - \mathrm{i}\frac{v_+ - v_-}{2}, \tag{4}$$

and a complex derivative

$$\frac{\mathbb{D}}{\mathrm{d}t} = \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}t_+} + \frac{\mathrm{d}}{\mathrm{d}t_-}\right) - \frac{\mathrm{i}}{2}\left(\frac{\mathrm{d}}{\mathrm{d}t_+} - \frac{\mathrm{d}}{\mathrm{d}t_-}\right) = \frac{\partial}{\partial t} + \mathcal{V}\cdot\nabla - \mathrm{i}\mathcal{D}\Delta. \tag{5}$$

This derivative operator is called the complex covariant derivative in the sense that it takes into account the effect of the fractal nature of space [3].

Using the bi-velocity $\mathcal{V}$ and the above complex operator, Nottale generalized the classical momentum $\boldsymbol{p}$ to a complex momentum $\mathcal{P} = m\mathcal{V}$, the real Lagrangian L to a complex one $\mathcal{L}$, with $\mathcal{P} = \partial\mathcal{L}/\partial\mathcal{V}$, and the classical action S to a complex action $\mathcal{S}$ so that $\mathcal{P} = \nabla\mathcal{S}$, or equivalently, $\mathcal{V} = \nabla\mathcal{S}/m$.

Finally, writing the complex action differently by introducing a complex-valued wavefunction $\psi = \mathrm{e}^{\mathrm{i}\mathcal{S}/\mathcal{S}_0}$, where $\mathcal{S}_0$ has dimensions of an action and taken to be $\hbar$, the bi-velocity may be written as [1]

$$\mathcal{V} = \frac{\mathcal{P}}{m} = \frac{\nabla\mathcal{S}}{m} = -\frac{\mathrm{i}\mathcal{S}_0}{m}\nabla\ln\psi = -\frac{\mathrm{i}\hbar}{m}(\nabla\psi)\psi^{-1}. \tag{6}$$

The Schrödinger equation governing the motion of a free particle of mass m is recovered, according to the principle of scale relativity [1, 2], by writing the classical equation of motion for the particle: $m\,\mathrm{d}\boldsymbol{v}/\mathrm{d}t = 0$, and making the two substitutions

$$\frac{\mathrm{d}}{\mathrm{d}t} \rightarrow \frac{\mathbb{D}}{\mathrm{d}t} \qquad \text{and} \qquad \boldsymbol{v} \rightarrow \mathcal{V}, \tag{7}$$

that yield

$$m\frac{\mathbb{D}}{\mathrm{d}t}\mathcal{V} = 0, \tag{8}$$

the corresponding 'equation of motion' in a nondifferentiable space. Inserting (5) and (6), one gets a third-order differential equation that, when integrated with $2m\mathcal{D} = \hbar$ [3], gives the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi + \frac{\hbar^2}{2m}\Delta\psi = 0. \tag{9}$$

For a given energy E, this equation writes in terms of the complex velocity $\mathcal{V}$ as [5, 10, 11]

$$\mathcal{V}^2 - 2\mathrm{i}\mathcal{D}\nabla\cdot\mathcal{V} - \frac{2E}{m} = 0. \tag{10}$$

This is the form taken by the stationary Schrödinger equation in scale relativity and the one that will be used in section 3 to drive the Pauli equation. To obtain the Klein–Gordon equation, Nottale assumed space *and* time to be fractal and nondifferentiable. As a consequence, (3)–(6) generalize as follows [2, 4, 7]:

$$\mathcal{V}^\mu = \frac{v_+^\mu + v_-^\mu}{2} - \mathrm{i}\frac{v_+^\mu - v_-^\mu}{2}, \tag{11}$$

$$\frac{\mathrm{d}f}{\mathrm{d}s_\pm} = \left(v_\pm^\mu\partial_\mu \mp \mathcal{K}\partial^\mu\partial_\mu\right)f, \tag{12}$$

$$\frac{\mathbb{D}}{\mathrm{d}s} = \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}s_+} + \frac{\mathrm{d}}{\mathrm{d}s_-}\right) - \frac{\mathrm{i}}{2}\left(\frac{\mathrm{d}}{\mathrm{d}s_+} - \frac{\mathrm{d}}{\mathrm{d}s_-}\right) = \mathcal{V}^\mu\partial_\mu + \mathrm{i}\mathcal{K}\partial^\mu\partial_\mu, \tag{13}$$

$$\mathcal{V}^\mu = \frac{\mathcal{P}^\mu}{mc} = \frac{\partial^\mu S}{mc} = \frac{\mathrm{i}\hbar}{mc}\frac{\partial^\mu\psi}{\psi} = \frac{\mathrm{i}\hbar}{mc}(\partial^\mu\psi)\psi^{-1}. \tag{14}$$

Here, $\mu = 1, \ldots, 4$.

The Klein–Gordon equation governing the motion of a free relativistic particle of mass m is recovered, according to the principle of scale relativity, by writing the relativistic equation of motion for the particle: $\mathrm{d}u^\alpha/\mathrm{d}s = 0$, and making the two substitutions

$$\frac{\mathrm{d}}{\mathrm{d}s} \rightarrow \frac{\mathbb{D}}{\mathrm{d}s} \qquad \text{and} \qquad \mathrm{u}^\alpha \rightarrow \mathcal{V}^\alpha, \tag{15}$$

that yield

$$\frac{\mathbb{D}}{\mathrm{d}s}\mathcal{V}^\alpha = 0, \tag{16}$$

the corresponding 'equation of motion' in a nondifferentiable spacetime. Inserting (13) and (14), one gets a third-order differential equation that, when integrated with $2mc\mathcal{K} = \hbar$, gives the Klein–Gordon equation:

$$\partial^\mu\partial_\mu\psi + \frac{m^2c^2}{\hbar^2}\psi = 0. \tag{17}$$

In [5], Pissondes showed that this equation may be written in terms of $\mathcal{V}^\mu$ in the form

$$\mathcal{V}^\mu\mathcal{V}_\mu + 2\mathrm{i}\mathcal{K}\partial^\mu\mathcal{V}_\mu - 1 = 0. \tag{18}$$

This form of the Klein–Gordon equation will be our starting point to get the Dirac equation in section 4.

3. The Pauli equation

Inspired by the work of Dirac in [12], Lévy-Leblond derived in [13] the Pauli equation by constructing a linear and first-order differential equation, which, when acted upon from the left by a linear operator gives back the Schrödinger second-order differential equation. Following a method similar to that of Lévy-Leblond but for different motivations, we shall arrive at the Pauli equation in the framework of scale relativity. Indeed, looking at the Schrödinger equation in its scale-relativistic form (10), we see that it has the form of the classical energy equation $v^2 - \frac{2E}{m} = 0$ but with an additional derivative term. In order to recover the classical form, we must first isolate the derivative operator by factorizing (10) to cast it into a product of a linear form in $\mathcal{V}$ and a linear operator acting from the left. We then take the linear form to vanish as a sufficient condition for the equation to hold. Let α^i $(i = 1, 2, 3)$, β and λ be five parameters to be determined, such that the following product gives back (10)

$$\left(\alpha^i\mathcal{V}^i - 2\mathrm{i}\mathcal{D}\alpha^i\nabla^i + \beta\frac{E}{m} + \lambda\right)\left(\alpha^j\mathcal{V}^j - \beta\frac{E}{m} - \lambda\right) = 0. \tag{19}$$

Working out the different factors, we see that we must have

$$\alpha^i\alpha^j\nabla^i\mathcal{V}^j = \nabla\cdot\mathcal{V}, \tag{20}$$

$$\beta^2 = 0, \qquad \lambda^2 = 0, \qquad \beta\frac{\lambda}{2} + \frac{\lambda}{2}\beta = 1, \tag{21}$$

$$\alpha^i\mathcal{V}^i\alpha^j\mathcal{V}^j = \mathcal{V}^2, \tag{22}$$

$$\left(\beta\frac{E}{m} + \lambda\right)\alpha^i\mathcal{V}^i - \alpha^i\mathcal{V}^i\left(\beta\frac{E}{m} + \lambda\right) = 0. \tag{23}$$

Using the fact that $\mathrm{rot}\mathcal{V} = 0$ [14], or equivalently, $\nabla^i\mathcal{V}^j - \nabla^j\mathcal{V}^i = 0$, (20) may be rewritten as

$$\tfrac{1}{2}\{\alpha^i, \alpha^j\}\nabla^i\mathcal{V}^j = \nabla\cdot\mathcal{V}, \tag{24}$$

where $\{\alpha^i, \alpha^j\}$ is the anticommutator of α^i and α^j. Hence, $\{\alpha^i, \alpha^j\} = 2\delta^{ij}$. We conclude that the three parameters α^i must be matrices rather than scalars and, more precisely, traceless matrices of even order. To satisfy the above anticommutation relation, the simplest choice is to use either the three Pauli σ^i matrices

$$\alpha^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \alpha^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \alpha^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{25}$$

or one of the two following groups of three 4×4 matrices constructed using the Pauli matrices:

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \qquad \text{or} \qquad \alpha^{\mathrm{i}} = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}. \tag{26}$$

From (21) we also conclude that β and λ cannot be scalars but matrices that might be chosen to be

$$\beta = \mathrm{i}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \lambda = -2\mathrm{i}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{27}$$

where 1 is either the unit number or the unit 2×2 matrix.

Now a sufficient condition for (19) to hold is

$$\alpha^i\mathcal{V}^i - \beta\frac{E}{m} - \lambda = 0. \tag{28}$$

However, this condition will be inconsistent with (22) if one uses (27) for β and λ and chooses for α^i either the Pauli matrices (25) or the first group of matrices in (26). This can easily be checked for the special case $\mathcal{V}^1 \neq 0$, and $\mathcal{V}^2 = \mathcal{V}^3 = 0$. So we take the second group of matrices in (26) to represent the α^i's and, as a consequence, $\mathcal{V}^i$, β and λ must also be 4×4 matrices. With these choices and (28), (23) becomes trivial whilst (22) yields

$$\mathcal{V}^2 = \left(\beta\frac{E}{m} + \lambda\right)^2 = \frac{2E}{m}\mathbf{1} \qquad \text{or} \qquad \frac{1}{2}\mathrm{m}\mathcal{V}^2 = \mathbf{E}, \tag{29}$$

where $\mathbf{E}$ is a diagonal 4×4 matrix with all entries equal to E. The above identity is nothing but the generalization in scale relativity of the classical energy equation $\frac{1}{2}mv^2 = E$. Thus, we see that, even though it is not possible using complex numbers to generalize $\frac{1}{2}mv^2 = E$ to $\frac{1}{2}m\mathcal{V}^2 = E$ [10], as it should according to the principle of scale relativity, which states that the form of the classical equations written in the ordinary space are preserved when transcribed into a fractal space via the substitutions (7), it is possible to satisfy this principle provided we substitute the matrices to complex numbers in the expression of $\mathcal{V}$.

Now using (6), (28) reads

$$\alpha^i \partial^i \psi - \frac{\mathrm{i}}{\hbar}(\beta E + \lambda m)\psi = 0, \tag{30}$$

implying that ψ must be a four-component spinor, which we write in the form $\psi = \binom{\varphi}{\eta}$, where φ and η are both two-spinors. Then, (26) and (27) give us

$$\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}\begin{pmatrix} \partial^i\varphi \\ \partial^i\eta \end{pmatrix} + \begin{pmatrix} 0 & -\frac{2m}{\hbar} \\ \frac{E}{\hbar} & 0 \end{pmatrix}\begin{pmatrix} \varphi \\ \eta \end{pmatrix} = 0. \tag{31}$$

Finally, since the energy E and the classical velocity $v^i = p^i/m$ generalize for a charge e in the presence of an electromagnetic field into $E - eU$ and $p^i/m - eA^i/mc$, respectively, the scale-relativistic energy E and velocity $\mathcal{V}^i$ of a charge e inside an electromagnetic field must also generalize, according to the principle of scale relativity, as

$$E \to E - eU \qquad \text{and} \qquad \mathcal{V}^i \to \frac{\mathcal{P}^i}{m} - \frac{e}{mc}A^i, \tag{32}$$

where U and A^i are, respectively, the scalar and the vector potentials. Thus, from (6), (26)–(28), (31) becomes

$$\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}\begin{pmatrix} \partial^i\varphi \\ \partial^i\eta \end{pmatrix} + \begin{pmatrix} -\frac{\mathrm{i}e}{\hbar c}\sigma^i A^i & -\frac{2m}{\hbar} \\ \frac{E-eU}{\hbar} & -\frac{\mathrm{i}e}{\hbar c}\sigma^i A^i \end{pmatrix}\begin{pmatrix} \varphi \\ \eta \end{pmatrix} = 0. \tag{33}$$

This is a system of two coupled equations in φ and η from which we can get a single equation in either φ or η alone in the form

$$\left\{E - eU - \frac{1}{2m}\left[\boldsymbol{\sigma}\cdot\left(\boldsymbol{P} - \frac{e}{c}\mathbf{A}\right)\right]\left[\boldsymbol{\sigma}\cdot\left(\boldsymbol{P} - \frac{e}{c}\boldsymbol{A}\right)\right]\right\}\varphi = 0, \tag{34}$$

where we have introduced the quantum mechanical notation $\boldsymbol{P} = -\mathrm{i}\hbar\nabla$, and $\boldsymbol{\sigma}$ and $\boldsymbol{A}$ are vectors whose components are σ^i and A^i, respectively.

Using the vector identity

$$(\boldsymbol{\sigma}\cdot\boldsymbol{b})(\boldsymbol{\sigma}\cdot\boldsymbol{c}) = \boldsymbol{b}\cdot\boldsymbol{c} + \mathrm{i}\boldsymbol{\sigma}\cdot(\boldsymbol{b}\times\boldsymbol{c}) \tag{35}$$

and

$$\boldsymbol{P}\times\boldsymbol{A} = -\mathrm{i}\hbar\nabla\times\boldsymbol{A} = -\mathrm{i}\hbar\boldsymbol{B}, \tag{36}$$

where $\boldsymbol{B}$ is the magnetic vector field, (34) becomes

$$\left\{E - eU - \frac{1}{2m}\left(\boldsymbol{P} - \frac{e}{c}\boldsymbol{A}\right)^2 + \frac{e\hbar}{2mc}\boldsymbol{\sigma}\cdot\boldsymbol{B}\right\}\varphi = 0, \tag{37}$$

which is the stationary Pauli equation containing the magnetic moment of the electron $e\hbar/2mc$.

4. The Dirac equation

Looking at the Klein–Gordon equation in its scale-relativistic form (18), we see that it has the form of the motion-relativistic invariant form $v^\mu v_\mu = 1$ but with an additional derivative term. Seeking to restore the classical form, in accordance with the principle of scale relativity that demands full covariance, we will isolate the derivative operator by a method similar to that used by Dirac to derive his equation. That is, we factorize the Klein–Gordon equation written in its scale-relativistic form (18) to cast it into a product of a linear form in $\mathcal{V}$ and a linear operator acting from the left. Then we take the linear form to vanish as a sufficient condition for the equation to hold. This, in turn, will lead us (as we shall see) to the Dirac equation. Let $\alpha_\mu(\mu = 1, \ldots, 4)$ and β be five parameters to be determined, such that the following product gives back (18)

$$(\alpha^\mu \mathcal{V}_\mu + 2\mathrm{i}\mathcal{K}\alpha^\mu\partial_\mu + \beta)(\alpha^\nu \mathcal{V}_\nu - \beta) = 0. \tag{38}$$

Working out the different factors, we see that we must have

$$\alpha^\mu\alpha^\nu\partial_\mu\mathcal{V}_\nu = \partial^\mu\mathcal{V}_\mu, \tag{39}$$

$$\beta^2 = 1, \tag{40}$$

$$\alpha^\mu\mathcal{V}_\mu\alpha^\nu\mathcal{V}_\nu = \mathcal{V}^\mu\mathcal{V}_\mu, \tag{41}$$

$$\beta\alpha^\mu\mathcal{V}_\mu - \alpha^\mu\mathcal{V}_\mu\beta = 0. \tag{42}$$

Using the fact that $\partial_\mu\mathcal{V}_\nu - \partial_\nu\mathcal{V}_\mu = 0$ [5], we rewrite (39) as

$$\tfrac{1}{2}\{\alpha^\mu, \alpha^\nu\}\partial_\mu\mathcal{V}_\nu = \partial^\mu\mathcal{V}_\mu. \tag{43}$$

Hence, $\{\alpha^\mu, \alpha^\nu\} = 2\eta^{\mu\nu}$. That is, the parameters α^μ cannot be pure numbers but matrices that satisfy the Dirac algebra, i.e., $\alpha^\mu \equiv \gamma^\mu$, where γ^μ are the familiar Dirac matrices. To satisfy (40) we may choose $\beta = \mathbf{1}$, where $\mathbf{1}$ is the unit 4×4 matrix. Then, the $\mathcal{V}^\mu$'s must also be 4×4 matrices.

Now a sufficient condition for (38) to hold is

$$\alpha^\mu\mathcal{V}_\mu - \beta = 0 \qquad \text{or} \qquad \gamma^\mu\mathcal{V}_\mu - \mathbf{1} = 0, \tag{44}$$

thanks to which (42) becomes trivial whilst (41) gives

$$\mathcal{V}^\mu\mathcal{V}_\mu = \mathbf{1}. \tag{45}$$

So again we see that, provided we substitute matrices to complex numbers in the expression of $\mathcal{V}^\mu$, the principle of scale relativity is realized since the quadratic invariant form $v^\mu v_\mu = 1$ is generalized to $\mathcal{V}^\mu\mathcal{V}_\mu = \mathbf{1}$ as it should and as it would if one uses the substitutions (15) to transcribe the classical invariant $v^\mu v_\mu$ to a fractal spacetime [4, 5].

Using (14) and (44) leads to

$$\frac{\mathrm{i}\hbar}{mc}\gamma^\mu(\partial_\mu\psi)\psi^{-1} - \mathbf{1} = 0 \qquad \text{or} \qquad \left(\frac{\mathrm{i}\hbar}{mc}\gamma^\mu\partial_\mu - \mathbf{1}\right)\psi = 0. \tag{46}$$

Since the γ^μ's are 4×4 matrices, ψ must be a four-component column vector, i.e., a Dirac spinor and the above equation is the free Dirac equation.

Finally, in scale relativity, the $\mathcal{V}^\mu$ of a charge e in the presence of an electromagnetic field is generalized into [15, 16]

$$\mathcal{V}^\mu = \frac{\mathcal{P}^\mu}{mc} - \frac{e}{mc^2}A^\mu, \tag{47}$$

where A^μ is the four-vector potential. Then, from (14) and (44), we get

$$\left[\gamma^\mu\left(\frac{\mathrm{i}\hbar}{mc}\partial_\mu - \frac{e}{mc^2}A_\mu\right) - \mathbf{1}\right]\psi = 0, \tag{48}$$

which is the Dirac equation in the presence of an electromagnetic field.

5. Concluding remarks

Our aim in the present paper was to derive both the Pauli and Dirac equations using the tools of scale relativity. The starting point of our derivations was the energy equations in scale relativity. Being of a different form from their classical counterparts by additional derivative terms, we have isolated the derivative operators to restore the classical forms and recovered in the process the familiar Pauli and Dirac equations. In [17], the classical form of the Hamiltonian $H = vp - L$, and therefore of all the energy equations, is recovered instead using a new velocity operator $\widehat{\mathcal{V}} = \mathcal{V} - \mathrm{i}\mathcal{D}\nabla$ [18] so that $\mathcal{H} = \widehat{\mathcal{V}}\mathcal{P} - \mathcal{L}$; the additional terms in the energy equations being maintained. In [5, 10, 11] these terms are interpreted as the quantum potentials of quantum mechanics and, as such, considered as manifestations of the nondifferentiability and fractality of spacetime. For that we should note here that our approach does not, as it may seem, contradict those results. Indeed, energy equations without derivative terms, and with them the Pauli and Dirac equations, steam from (28) and (44), respectively. These, as we saw, are only sufficient and not necessary conditions for our factorizations (19) and (38) to hold. This means that the suppressions of the additional terms occur only when (28) or (44) are satisfied in which case we get spinors or equivalently fermions.

Although new concepts were needed for the derivations of the Pauli and Dirac equations in [6–8], these had the advantage of proposing a geometric origin for the spin in quantum mechanics. In this respect, our approach is complementary to those. Our method, however, permits not only to derive both equations without additional concepts other than those used to get the Schrödinger and Klein–Gordon equations in scale relativity but also to give a physical meaning to their standard derivations in quantum mechanics. Indeed, before the work of Lévy-Leblonc, it was believed that the emergence of the spin from the Dirac equation was purely relativistic. The derivation by Lévy-Leblonc of the Pauli equation with the correct gyromagnetic moment of the electron from the nonrelativistic Schrödinger equation without passing through the Dirac equation showed that the electron spin was rather a consequence of the 'postulate' of the linearization of the wave equations [19]. Here, we have seen that the physical motivation behind the factorization that actually leads to this linearization is the principle of scale relativity that imposes on physical equations to keep their classical forms in scale relativity.

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